METRIC HEIGHTS ON AN ABELIAN GROUP

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Abstract. Suppose $m(\alpha)$ denotes the Mahler measure of the non-zero algebraic number α. For each positive real number t, the author studied a version $m_t(\alpha)$ of the Mahler measure that has the triangle inequality. The construction of m_t is generic, and may be applied to a broader class of functions defined on any Abelian group G. We prove analogs of known results with an abstract function on G in place of the Mahler measure. In the process, we resolve an earlier open problem stated by the author regarding $m_t(\alpha)$.

1. Heights and their metric versions

Suppose that K is a number field and v is a place of K dividing the place p of $\mathbb{Q}$. Let K_v and $\mathbb{Q}_p$ be their respective completions so that K_v is a finite extension of $\mathbb{Q}_p$. We note the well-known fact that

$$\sum_{v|p}[K_v:\mathbb{Q}_p]=[K:\mathbb{Q}],$$

where the sum is taken over all places v of K dividing p. Given $x\in K_v$, we define $\|x\|_v$ to be the unique extension of the p-adic absolute value on $\mathbb{Q}_p$ and set

$$|x|_v=\|x\|_v^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}. \tag{1.1}$$

If $\alpha\in K$, then $\alpha\in K_v$ for every place v, so we may define the *(logarithmic) Weil height* by

$$h(\alpha)=\sum_v \log^+|\alpha|_v.$$

Due to our normalization of absolute values (1.1), this definition is independent of K, meaning that h is well-defined as a function on the multiplicative group $\overline{\mathbb{Q}}^\times$ of non-zero algebraic numbers.

It follows from Kronecker's Theorem that $h(\alpha)=0$ if and only if α is a root of unity, and it can easily be verified that $h(\alpha^n)=|n|\cdot h(\alpha)$ for all integers n. In particular, we see that $h(\alpha)=h(\alpha^{-1})$. A theorem of Northcott [9] asserts that, given a positive real number D, there are only finitely many algebraic numbers α with $\deg\alpha\le D$ and $h(\alpha)\le D$.

The Weil height is closely connected to a famous 1933 problem of D.H. Lehmer [7]. The *(logarithmic) Mahler measure* of a non-zero algebraic number α is defined by

$$m(\alpha)=\deg\alpha\cdot h(\alpha). \tag{1.2}$$

In attempting to construct large prime numbers, Lehmer came across the problem of determining whether there exists a sequence of algebraic numbers $\{\alpha_n\}$, not

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roots of unity, such that $m(\alpha_n)$ tends to 0 as $n \to \infty$. This problem remains unresolved, although substantial evidence suggests that no such sequence exists (see [1, 8, 13, 14], for instance). This assertion is typically called Lehmer's conjecture.

Conjecture (Lehmer's Conjecture)**.** There exists $c > 0$ such that $m(\alpha) \geq c$ whenever $\alpha \in \overline{\mathbb{Q}}^\times$ is not a root of unity.

Dobrowolski [2] provided the best known lower bound on $m(\alpha)$ in terms of $\deg \alpha$, while Voutier [15] later gave a version of this result with an effective constant. Nevertheless, only little progress has been made on Lehmer's conjecture for an arbitrary algebraic number α.

Dubickas and Smyth [3, 4] were the first to study a modified version of the Mahler measure that has the triangle inequality. They defined the *metric Mahler measure* by

$$m_1(\alpha) = \inf\left\{ \sum_{n=1}^N m(\alpha_n) : N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^\times\ \alpha = \prod_{n=1}^N \alpha_n \right\},$$

so that the infimum is taken over all ways of writing α as a product of algebraic numbers. It is easily verified that $m_1(\alpha\beta) \leq m_1(\alpha) + m_1(\beta)$, and that m_1 is well-defined on $\overline{\mathbb{Q}}^\times/\mathrm{Tor}(\overline{\mathbb{Q}}^\times)$. It is further noted in [4] that $m_1(\alpha) = 0$ if and only if α is a torsion point of $\overline{\mathbb{Q}}^\times$ and that $m_1(\alpha) = m_1(\alpha^{-1})$ for all $\alpha \in \overline{\mathbb{Q}}^\times$. These facts ensure that $(\alpha, \beta) \mapsto m_1(\alpha\beta^{-1})$ defines a metric on $\overline{\mathbb{Q}}^\times/\mathrm{Tor}(\overline{\mathbb{Q}}^\times)$. This metric induces the discrete topology if and only if Lehmer's conjecture is true.

The author [11, 12] further extended this definition to define the *t-metric Mahler measure*

$$m_t(\alpha) = \inf\left\{ \left(\sum_{n=1}^N m(\alpha_n)^t \right)^{1/t} : N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^\times\ \alpha = \prod_{n=1}^N \alpha_n \right\}. \tag{1.3}$$

In this context, we examined the functions $t \mapsto m_t(\alpha)$ for a fixed algebraic number α. It is shown, for example, that this is a continuous piecewise function where each piece is an L_p norm of a real vector.

Although (1.3) may be useful in studying Lehmer's problem, this construction applies in far greater generality. The natural abstraction of (1.3) is examined by the author in [11], although we only present the most basic results that are needed in studying the t-metric Mahler measures. The goal of this article is to recover some results of [11] and [12] with an abstract height function in place of the Mahler measure. In the process, we shall uncover new results regarding the t-metric Mahler measure, including the resolution of a problem posed in [12]. These results are reported in Section 2.

Before we can state our main theorem, we must recall the basic definitions and results of [11]. Let G be a multiplicatively written Abelian group. We say that $\phi : G \to [0, \infty)$ is a *(logarithmic) height* on G if

(i) $\phi(1) = 0$, and
(ii) $\phi(\alpha) = \phi(\alpha^{-1})$ for all $\alpha \in G$.

It is well-known that both the Weil height and the Mahler measure are heights on $\overline{\mathbb{Q}}^\times$. If t is a positive real number then we say that ϕ has the *t-triangle inequality* if

$$\phi(\alpha\beta) \leq \left(\phi(\alpha)^t + \phi(\beta)^t\right)^{1/t}$$

for all $\alpha, \beta \in G$. We say that ϕ has the ∞-*triangle inequality* if

$$\phi(\alpha\beta) \leq \max\{\phi(\alpha), \phi(\beta)\}$$

for all $\alpha, \beta \in G$. For appropriate t, we say that these functions are t-*metric heights.* We observe that the 1-triangle inequality is simply the classical triangle inequality while the ∞-triangle inequality is the strong triangle inequality. If $s \geq t$ and ϕ is an s-metric height then it is also a t-metric height. The metric height properties also yield some compatibility of ϕ with the group structure of G.

Proposition 1.1. *If $\phi : G \to [0, \infty)$ is a t-metric height for some $t \in (0, \infty]$ then*

(i) $\phi^{-1}(0)$ is a subgroup of G.
(ii) $\phi(\zeta\alpha) = \phi(\alpha)$ for all $\alpha \in G$ and $\zeta \in \phi^{-1}(0)$. That is, ϕ is well-defined on the quotient $G/\phi^{-1}(0)$.
(iii) If $t \geq 1$, then the map $(\alpha, \beta) \mapsto \phi(\alpha\beta^{-1})$ defines a metric on $G/\phi^{-1}(0)$.

If we are given an arbitrary height ϕ on G and $t > 0$, then following (1.3), we obtain a natural t-metric height associated to ϕ. Given any subset $S \subseteq G$ containing the identity e, we write

$$S^\infty = \{(\alpha_1, \alpha_2, \ldots) : \alpha_n \in S,\ \alpha_n = e \text{ for all but finitely many } n\}.$$

If S is a subgroup, then S^∞ is also a group by applying the operation of G coordinatewise. Define the group homomorphism $\tau_G : G^\infty \to G$ by $\tau_G(\alpha_1, \alpha_2, \cdots) = \prod_{n=1}^\infty \alpha_n$. For each point $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$, we write the L_t norm of $\mathbf{x}$

$$\|\mathbf{x}\|_t = \begin{cases} \left(\sum_{n=1}^\infty |x_n|^t\right)^{1/t} & \text{if } t \in (0, \infty) \\ \max_{n\geq 1}\{|x_n|\} & \text{if } t = \infty \end{cases}$$

The *t-metric version of ϕ* is the map $\phi_t : G \to [0, \infty)$ given by

$$\phi_t(\alpha) = \inf\left\{\|(\phi(\alpha_1), \phi(\alpha_2), \ldots)\|_t : (\alpha_1, \alpha_2, \ldots) \in G^\infty \text{ and } \tau_G(\alpha_1, \alpha_2, \ldots) = \alpha\right\}.$$

Alternatively, we could write

$$\phi_t(\alpha) = \inf\left\{\left(\sum_{n=1}^N \phi(\alpha_n)^t\right)^{1/t} : N \in \mathbb{N},\ \alpha_n \in G \text{ and } \alpha = \prod_{n=1}^N \alpha_n\right\}$$

for $t \in (0, \infty)$ and

$$\phi_\infty(\alpha) = \inf\left\{\max_{1\leq n\leq N}\{\phi(\alpha_n)\} : N \in \mathbb{N},\ \alpha_n \in G \text{ and } \alpha = \prod_{n=1}^N \alpha_n\right\}.$$

Among other things, we see that ϕ_t is indeed a t-metric height on G.

Proposition 1.2. *If $\phi : G \to [0, \infty)$ is a height on G and $t \in (0, \infty]$ then*

(i) ϕ_t is a t-metric height on G with $\phi_t \leq \phi$.
(ii) If ψ is a t-metric height with $\psi \leq \phi$ then $\psi \leq \phi_t$.
(iii) $\phi = \phi_t$ if and only if ϕ is a t-metric height. In particular, $(\phi_t)_t = \phi_t$.
(iv) If $s \in (0, t]$ then $\phi_s \geq \phi_t$.

2. The function $t \mapsto \phi_t(\alpha)$

For the remainder of this article, we shall assume that ϕ is a height on the Abelian group G and that α is a fixed element of G. All subsequent definitions depend on these choices, although we will often suppress this dependency in our notation. We say that a set $S \subseteq G$ containing the identity *replaces G at t* if

$$\phi_t(\alpha) = \inf \left\{ \|(\phi(\alpha_1), \phi(\alpha_2), \ldots)\|_t : (\alpha_1, \alpha_2, \ldots) \in S^\infty \text{ and } \tau_G(\alpha_1, \alpha_2, \ldots) = \alpha \right\}.$$

In other words, we need only consider points in S in the definition of $\phi_t(\alpha)$.

For a given height ϕ, it is standard to ask whether the infimum in its definition is attained. Before proceeding further, we give two equivalent conditions.

Theorem 2.1. *If $t \in (0, \infty]$ then the following conditions are equivalent.*

(i) The infimum in the definition of $\phi_t(\alpha)$ is attained.
(ii) There exists a finite set R that replaces G at t.
(iii) There exists a set S, with $\phi(S)$ finite, that replaces G at t.

We now wish to study the behavior of $f_{\phi,\alpha} : (0, \infty) \to [0, \infty)$, defined by

$$f_{\phi,\alpha}(t) = \phi_t(\alpha),$$

on a given open interval $I \subseteq (0, \infty)$ as was done with the Mahler measure in [11, 12]. We say that $S \subseteq G$ *replaces G uniformly on I* if S replaces G at all $t \in I$. In this case, it is important to note that S is independent of t. We say that a subset $K \subseteq I$ is *uniform* if there exists a point $\mathbf{x} \in \mathbb{R}^\infty$ such that $f_{\phi,\alpha}(t) = \|\mathbf{x}\|_t$ for all $t \in K$. Indeed, the uniform subintervals of I are the simplest to understand as there always exist $x_1, \ldots, x_N \in \mathbb{R}$ such that

$$f_{\phi,\alpha}(t) = \left(\sum_{n=1}^{N} |x_n|^t \right)^{1/t}$$

for all $t \in K$. We say that t is *standard* if there exists a uniform open interval $J \subseteq I$ containing t. Otherwise, we say that t is *exceptional.* The author asked in [12] whether the Mahler measure has only finitely many exceptional points. We answer this question in the affirmative and prove several additional facts about heights in general.

Theorem 2.2. *Assume $I \subseteq (0, \infty)$ is an open interval such that the infimum in $\phi_t(\alpha)$ is attained for all $t \in I$. If G is a countable group then the following conditions are equivalent.*

(i) There exists a finite set $\mathcal{X} \subseteq \mathbb{R}^\infty$ such that $\phi_t(\alpha) = \min\{\|\mathbf{x}\|_t : \mathbf{x} \in \mathcal{X}\}$ for all $t \in I$.
(ii) I contains only finitely many exceptional points.
(iii) There exists a finite set R that replaces G uniformly on I.
(iv) There exists a set S, with $\phi(S)$ finite, that replaces G uniformly on I.

In the case where ϕ is the Mahler measure, and where I is a bounded interval, the majority of Theorem 2.2 was established in [12]. Indeed, in this special case, we showed that (iv) $\implies$ (i) $\iff$ (ii). Nevertheless, Theorem 2.2 is considerably more general than any earlier work because it requires neither the assumption that ϕ is the Mahler measure nor the assumption that I is bounded. Moreover, since the Mahler measure is known to satisfy (iv), we have now resolved the aforementioned question of [12].

Corollary 2.3. *The Mahler measure m on $\overline{\mathbb{Q}}^\times$ has only finitely many exceptional points in* $(0,\infty)$.

This means that $f_{m,\alpha}$ is a piecewise function, with finitely many pieces, where each piece is the L_t norm of a vector with real entries. Moreover, the infimum in $m_t(\alpha)$ is attained by a single point for all sufficiently large t. In the case where $\alpha \in \mathbb{Z}$, it follows from [6] that this point may be taken to be the vector having the prime factors of α as its entries.

Still considering the case $\phi = m$, the results of [11] show that

$$S_\alpha = \left\{ \gamma \in \overline{\mathbb{Q}}^\times : \gamma^n \in K_\alpha \text{ for some } n \in \mathbb{N},\ m(\gamma) \leq m(\alpha) \right\},$$

where K_α is the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$, replaces $\overline{\mathbb{Q}}^\times$ uniformly on $(0,\infty)$. However, it is well-known that $m(S_\alpha)$ is finite, so Theorem 2.2 implies the existence of a finite set R_α that replaces $\overline{\mathbb{Q}}^\times$ uniformly on $(0,\infty)$. It remains open to determine such a set, although we suspect that

$$R_\alpha = \left\{ \gamma \in \overline{\mathbb{Q}}^\times : \deg(\gamma) \leq \deg \alpha,\ m(\gamma) \leq m(\alpha) \right\}$$

satisfies this property. The work of Jankauskas and the author [6] provides examples, however, in which K_α does not replace $\overline{\mathbb{Q}}^\times$ uniformly on $(0,\infty)$.

Returning to Theorem 2.2 and taking an arbitrary ϕ, it is important to note the necessity of our assumption that the infimum in $\phi_t(\alpha)$ is attained for every $t \in I$. Indeed, Theorem 1.6 of [11] asserts that

$$h_t(\alpha) = \begin{cases} h(\alpha) & \text{if } t \leq 1 \\ 0 & \text{if } t > 1, \end{cases}$$

where h denotes the logarithmic Weil height on $\overline{\mathbb{Q}}^\times$. The intervals $(0,1)$ and $(1,\infty)$ are both uniform by using $\mathbf{x} = (h(\alpha), 0, 0, \ldots)$ and $\mathbf{x} = (0,0,0,\ldots)$, respectively, so 1 is the only possible exceptional point. However, $t \mapsto h_t(\alpha)$ is discontinuous on $(0,2)$ whenever α is not a root of unity, so condition (i) does not hold on this interval. Theorem 2.2 does not apply because the infimum in $h_t(\alpha)$ is not attained for any $t > 1$. Nevertheless, the assumption (iv) would imply that the infimum in $\phi_t(\alpha)$ is attained for all $t \in I$ from Theorem 2.1.

The assumption that G is countable is needed only to prove that (ii) $\implies$ (iii). If an interval I is uniform, then $\phi_t(\alpha) = \|\mathbf{x}\|_t$ for all $t \in I$. It seems plausible, however, that $\mathbf{x}$ does not arise from a point that attains the infimum in $\phi_t(\alpha)$. This concern can be resolved if, for example, G is countable, although we do not know whether this assumption is necessary.

3. Proofs

Our first lemma is the primary component in the proof of Theorem 2.1 and will also be used in the proof of Theorem 2.2.

Lemma 3.1. *Suppose $J \subseteq (0,\infty)$ is bounded. If there exists a set $S \subseteq G$, with $\phi(S)$ finite, such that S replaces G uniformly on J, then there exists a finite set $D \subseteq \tau_G^{-1}(\alpha) \cap S^\infty$ such that*

$$\phi_t(\alpha) = \min \left\{ \left(\sum_{n=1}^\infty \phi(\alpha_n)^t \right)^{1/t} : (\alpha_1, \alpha_2 \ldots) \in D \right\}$$

for all $t \in J$. In particular, there exists a finite set $\mathcal{X} \subseteq \mathbb{R}^\infty$ such that $\phi_t(\alpha) = \min\{\|\mathbf{x}\|_t : \mathbf{x} \in \mathcal{X}\}$ for all $t \in J$.

Proof. We know that that $\phi_t(\alpha)$ is the infimum of

$$\left(\sum_{n=1}^{\infty} \phi(\alpha_n)^t\right)^{1/t} \tag{3.1}$$

over all points $(\alpha_1, \alpha_2, \ldots) \in G^\infty$ satisfying the following three conditions:
(a) $\alpha = \prod_{n=1}^\infty \alpha_n$.
(b) $\alpha_n \in S$ for all n.
(c) $\left(\sum_{n=1}^\infty \phi(\alpha_n)^t\right)^{1/t} \leq \phi(\alpha)$.
If $(\alpha_1, \alpha_2, \ldots) \in G^\infty$ is a point satisfying these conditions, let B be the number of its coordinates that do not belong to $\phi^{-1}(0)$. Also set $\delta = \inf \phi(S \setminus \phi^{-1}(0))$. Since $\phi(S)$ is finite, we know that $\delta > 0$. By property (c), we see that

$$\delta^t \cdot B \leq \sum_{n=1}^{\infty} \phi(\alpha_n)^t \leq \phi(\alpha)^t,$$

so it follows that $B \leq \phi(\alpha)^u/\delta^u$, where u is any upper bound for J. Therefore, at most $\phi(\alpha)^u/\delta^u$ terms $\phi(\alpha_n)$ in (3.1) can be non-zero and the result follows. $\square$

In view of this lemma, the proof of Theorem 2.1 is essentially finished. Indeed, one obtains (iii) $\implies$ (i) immediately from the lemma by taking $J = \{t\}$, while the other implications of the theorem are obvious.

We shall now proceed with the proof of Theorem 2.2 which will require three additional lemmas, the first of which is a standard trick from complex analysis.

Lemma 3.2. *If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\infty$ and $\|\mathbf{x}\|_t = \|\mathbf{y}\|_t$ for all t on a set having a limit point in $\mathbb{R}$, then $\|\mathbf{x}\|_t = \|\mathbf{y}\|_t$ for all $t \in (0, \infty)$.*

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots, y_M, 0, 0, \ldots)$, $g_{\mathbf{x}}(t) = \|\mathbf{x}\|_t^t$ and $g_{\mathbf{y}}(t) = \|\mathbf{y}\|_t^t$. Therefore,

$$g_{\mathbf{x}}(z) = \sum_{n=1}^{N} |x_n|^z \quad \text{and} \quad g_{\mathbf{y}}(t) = \sum_{m=1}^{M} |y_m|^z$$

are entire functions. Hence, if they agree on a set having a limit point in $\mathbb{C}$, then they are equal in $\mathbb{C}$.

$\square$

We also must study the behavior of intervals in which every point is standard. While every point in such an interval is guaranteed only to have a uniform open neighborhood, it turns out that this neighborhood may be taken to be the interval itself.

Lemma 3.3. *Suppose $0 \leq a < b \leq \infty$. Then (a, b) is uniform if and only if every point in (a, b) is standard.*

Proof. If (a, b) is uniform, it is obvious that every point in (a, b) is standard. So we will assume that every point in (a, b) is standard and that (a, b) is not uniform. Let $t_0 \in (a, b)$ so there exists $\varepsilon > 0$ and $\mathbf{x} \in \mathbb{R}$ such that $\phi_t(\alpha) = \|\mathbf{x}\|_t$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. We have assumed that (a, b) is not uniform, so there must exist $s_0 \in (a, b)$ such that $\phi_{s_0}(\alpha) \neq \|\mathbf{x}\|_{s_0}$. Clearly $s_0 \neq t_0$.

Assume without loss of generality that $s_0 > t_0$ and let

$$u = \inf\{t \in [t_0, b) : \phi_t(\alpha) \neq \|\mathbf{x}\|_t\}.$$

We clearly have that $u \in [t_0 + \varepsilon, s_0] \subseteq (a, b)$, meaning, in particular, that u is standard. Therefore, there exists a neighborhood $(a_0, b_0) \subseteq (t_0, b)$ of u and $\mathbf{y} \in \mathbb{R}^\infty$ such that

$$\phi_t(\alpha) = \|\mathbf{y}\|_t \tag{3.2}$$

for all $t \in (a_0, b_0)$. However, by definition of u, we also know that $\phi_t(\alpha) = \|\mathbf{x}\|_t$ for all $t \in (a_0, u)$. Therefore, $\|\mathbf{y}\|_t$ and $\|\mathbf{x}\|_t$ agree on a set having a limit point in $\mathbb{R}$, and we may apply Lemma 3.2 to find that they agree on (a, b). The definition of u further implies the existence of a point $s \in [u, b_0)$ such that $\phi_s(\alpha) \neq \|\mathbf{x}\|_s$. Combining this with (3.2), we see that $\|\mathbf{x}\|_s \neq \|\mathbf{y}\|_s$, a contradiction. □

For a uniform interval I, we always know there exists $\mathbf{x} \in \mathbb{R}^\infty$ such that $\phi_t(\alpha) = \|\mathbf{x}\|_t$ for all $t \in I$. Nevertheless, it is possible that $\mathbf{x}$ cannot be chosen to arise from a point that attains the infimum in $\phi_t(\alpha)$. The following lemma provides an adequate resolution to this problem in the case where G is countable.

Lemma 3.4. *Suppose G is a countable group and $K \subseteq (0, \infty)$ is an uncountable subset. Assume that the infimum in $\phi_t(\alpha)$ is attained for all $t \in K$. Then K is uniform if and only if there exists $(\alpha_1, \alpha_2, \ldots) \in G^\infty$ such that*

$$\alpha = \prod_{n=1}^{\infty} \alpha_n \quad \text{and} \quad \phi_t(\alpha) = \left(\sum_{n=1}^{\infty} \phi(\alpha_n)^t\right)^{1/t} \tag{3.3}$$

for all $t \in K$.

Proof. Assuming (3.3), it is obvious that K is uniform. Hence, we assume that K is uniform and let $\mathbf{x} \in \mathbb{R}^\infty$ be such that

$$\phi_t(\alpha) = \|\mathbf{x}\|_t \tag{3.4}$$

for all $t \in K$. We have assumed the infimum in $\phi_t(\alpha)$ is attained for all $t \in K$. Hence, for each such t, we may select $(\alpha_{t,1}, \alpha_{t,2}, \ldots) \in G^\infty$ such that

$$\alpha = \prod_{n=1}^{\infty} \alpha_{t,n} \quad \text{and} \quad \phi_t(\alpha) = \left(\sum_{n=1}^{\infty} \phi(\alpha_{t,n})^t\right)^{1/t}.$$

Since K is uncountable, $t \mapsto (\alpha_{t,1}, \alpha_{t,2}, \ldots)$ maps an uncountable set to a countable set. In particular, this map must be constant on an uncountable subset $J \subseteq K$. Hence, there exists $(\alpha_1, \alpha_2, \ldots) \in G^\infty$ such that

$$\alpha = \prod_{n=1}^{\infty} \alpha_n \quad \text{and} \quad \phi_t(\alpha) = \left(\sum_{n=1}^{\infty} \phi(\alpha_n)^t\right)^{1/t},$$

for all $t \in J$. Applying (3.4), we have that

$$\left(\sum_{n=1}^{\infty} \phi(\alpha_n)^t\right)^{1/t} = \|\mathbf{x}\|_t \tag{3.5}$$

for all $t \in J$. Since J is uncountable, it must contain a limit point in $[0, \infty)$, and it follows from Lemma 3.2 that (3.5) holds for all $t \in K$. The lemma now follows from (3.4). $\square$

Before proceeding with the proof of Theorem 2.2, we provide a definition that will simplify the proof's language. If $\mathcal{X} \subseteq \mathbb{R}^\infty$, we say that $s \in (0, \infty)$ is an *intersection point of* $\mathcal{X}$ if there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ such that $\|\mathbf{x}\|_s = \|\mathbf{y}\|_s$ but $t \mapsto \|\mathbf{x}\|_t$ is not the same function as $t \mapsto \|\mathbf{y}\|_t$.

Proof of Theorem 2.2. We begin by proving that (i) $\implies$ (ii). We first show that $\mathcal{X}$ has only finitely many intersection points. To see this, assume that

$$(x_1, x_2, \ldots, x_N, 0, 0, \ldots), (y_1, y_2, \ldots, y_M, 0, 0, \ldots) \in \mathcal{X},$$

with $x_n, y_m \neq 0$, are distinct elements and define

$$F(t) = \|\mathbf{x}\|_t^t - \|\mathbf{y}\|_t^t = \sum_{n=1}^{N} |x_n|^t - \sum_{m=1}^{M} |y_n|^t.$$

We may assume without loss of generality that

$$F(t) = \sum_{k=1}^{K} a_k b_k^t$$

for some positive integer K and nonzero real numbers a_k and b_k with $b_1 > \cdots > b_K > 0$. Then it follows that

$$\frac{F(t)}{a_1 b_1^t} = 1 + \sum_{k=2}^{K} \frac{a_k}{a_1} \left(\frac{b_k}{b_1} \right)^t$$

which tends to 1 as $t \to \infty$. Since $a_1 b_1^t$ has no zeros, all zeros of $F(t)$ must lie in a bounded subset of $(0, \infty)$. Viewing t as a complex variable, $F(t)$ is an entire function which is not identically 0, so it cannot have infinitely many zeros in a bounded set. We conclude that there are only finitely many points t such that $\|\mathbf{x}\|_t = \|\mathbf{y}\|_t$. Since $\mathcal{X}$ is finite, it can have only finitely many intersection points.

It is now enough to show that every exceptional point is also an intersection point of $\mathcal{X}$. We will prove the contrapositive of this statement, so assume that $t \in I$ is not an intersection point of $\mathcal{X}$. Since there are only finitely many intersection points, we know there exists an open interval $J \subseteq I$ containing t and having no intersection points of $\mathcal{X}$.

Now assume that J fails to be uniform and fix $t \in J$. By our assumption, we know there exists $\mathbf{x} \in \mathcal{X}$ such that $\phi_t(\alpha) = \|\mathbf{x}\|_t$. There must also exist $s \in J$ such that $\phi_s(\alpha) \neq \|\mathbf{x}\|_s$. Since $\mathbf{x} \in \mathcal{X}$, we certainly have that $\phi_s(\alpha) < \|\mathbf{x}\|_s$. On the other hand, there must exist $\mathbf{y} \in \mathcal{X}$ such that $\phi_s(\alpha) = \|\mathbf{y}\|_s$, and we note that $\phi_t(\alpha) \leq \|\mathbf{y}\|_t$. Therefore, we have shown that

$$\|\mathbf{x}\|_t \leq \|\mathbf{y}\|_t \quad \text{and} \quad \|\mathbf{x}\|_s > \|\mathbf{y}\|_s.$$

By the Intermediate Value Theorem, there exists a point r between s and t such that $\|\mathbf{x}\|_r = \|\mathbf{y}\|_r$. This means that J contains an intersection point, contradicting our assumption that J contains none. Therefore, we conclude that J is uniform, implying that t is standard. Indeed, we have now shown that every exceptional point is also an intersection point of $\mathcal{X}$.

We now prove that (ii) $\implies$ (iii). To see this, assume that $I = (t_0, t_M)$ and that the exceptional points in I are given by

$$t_1 < t_2 < \cdots < t_{M-1}.$$

For each integer $m \in [1, M]$, it follows that (t_{m-1}, t_m) contains only standard points. Applying Lemma 3.3, we conclude that all of these intervals are uniform.

Now we apply Lemma 3.4 with (t_{m-1}, t_m) in place of K. For each integer $m \in [1, M]$, there must exist $(\alpha_{m,1}, \alpha_{m,2}, \ldots) \in G^\infty$ such that

$$\alpha = \prod_{n=1}^{\infty} \alpha_{m,n} \quad \text{and} \quad \phi_t(\alpha) = \left(\sum_{n=1}^{\infty} \phi(\alpha_{m,n})^t \right)^{1/t}, \tag{3.6}$$

for all $t \in (t_{m-1}, t_m)$. Moreover, for each integer $m \in [1, M)$, we select $(\beta_{m,1}, \beta_{m,2}, \ldots) \in G^\infty$ such that

$$\alpha = \prod_{n=1}^{\infty} \beta_{m,n} \quad \text{and} \quad \phi_{t_m}(\alpha) = \left(\sum_{n=1}^{\infty} \phi(\beta_{m,n})^{t_m} \right)^{1/t_m}.$$

Now let

$$R = \{\alpha_{m,n} : 1 \leq m \leq M,\ n \in \mathbb{N}\} \cup \{\beta_{m,n} : 1 \leq m < M,\ n \in \mathbb{N}\}$$

and note that R is clearly finite.

To see that R replaces G uniformly on I, we must assume that $t \in I$. If $t \in (t_{m-1}, t_m)$ for some m, then

$$\begin{aligned} \phi_t(\alpha) &\leq \inf \left\{ \|(\phi(\alpha_1), \phi(\alpha_2), \ldots)\|_t : (\alpha_1, \alpha_2, \ldots) \in R^\infty,\ \alpha = \prod_{n=1}^{\infty} \alpha_n \right\} \\ &\leq \|(\phi(\alpha_{m,1}), \phi(\alpha_{m,2}), \ldots)\|_t \\ &= \phi_t(\alpha), \end{aligned}$$

where the last equality is exactly the right hand side of (3.6). Given any m, we have now shown that

$$\phi_t(\alpha) = \inf \left\{ \|(\phi(\alpha_1), \phi(\alpha_2), \ldots)\|_t : (\alpha_1, \alpha_2, \ldots) \in R^\infty,\ \alpha = \prod_{n=1}^{\infty} \alpha_n \right\} \tag{3.7}$$

for all $t \in (t_{m-1}, t_m)$. Moreover, the same argument shows that (3.7) holds when $t = t_m$ for some $1 \leq m < M$. We finally obtain that (3.7) holds for all $t \in I$ as required.

It is immediately obvious that (iii) $\implies$ (iv), so to complete the proof, we must now show that (iv) $\implies$ (i). Lemma 3.1 establishes this implication in the case where I is a bounded interval, so we shall assume that $I = (a, \infty)$ for some $a \geq 0$. Moreover, this lemma enables us to assume the existence of a set $A \subseteq \phi(S)^\infty$ such that

$$\phi_t(\alpha) = \min \{\|\mathbf{x}\|_t : \mathbf{x} \in A\}$$

for all $t \in (a, \infty)$. Without loss of generality, we may assume that

$$x_1 \geq x_2 \geq x_3 \geq \ldots$$

for all $(x_1, x_2, \ldots) \in A$. Then we define a recursive sequence of sets $A_k \subseteq A$ in the following way.

(i) Let $A_0 = A$.

(ii) Given A_k, let $M_{k+1} = \min\{x_{k+1} : (x_1, x_2, \ldots) \in A_k\}$. Observe that M_{k+1} exists and is non-negative because $\phi(S)$ is a finite set of non-negative numbers and $x_{k+1} \in \phi(S)$. Now let $A_{k+1} = \{(x_1, x_2, \ldots) \in A_k : x_{k+1} = M_{k+1}\}$.

It is immediately clear that all of these sets are nonempty and $A_{k+1} \subseteq A_k$ for all k.

We claim that for every $k \geq 0$, there exists $s_k \in [a, \infty)$ such that

$$\phi_t(\alpha) = \min\{\|\mathbf{x}\|_t : \mathbf{x} \in A_k\} \tag{3.8}$$

for all $t \in (s_k, \infty)$. We prove this assertion using induction on k, and we obtain the base case easily by taking $s_0 = a$.

Now assume that there exists $s_k \in [a, \infty)$ such that (3.8) holds for all $t \in (s_k, \infty)$. If $A_{k+1} = A_k$, then we take $s_{k+1} = s_k$ and the result follows. Assuming otherwise, we may define

$$M'_{k+1} = \min\{x_{k+1} : (x_1, x_2, \ldots) \in A_k \setminus A_{k+1}\}.$$

As before, we know that this minimum exists because x_{k+1} belongs to the finite set $\phi(S)$ for all $(x_1, x_2, \ldots) \in A_k \setminus A_{k+1}$. Furthermore, we plainly have that $M'_{k+1} > M_{k+1}$. If $t > s_k$ then (3.8) implies that

$$\phi_t(\alpha)^t = \min\{\|\mathbf{x}\|_t^t : \mathbf{x} \in A_k\} = \min\left\{\sum_{n=1}^{\infty} x_n^t : (x_1, x_2, \ldots) \in A_k\right\}.$$

Then we conclude that

$$\phi_t(\alpha)^t - \sum_{n=1}^{k} M_n^t = \min\left\{\sum_{n=k+1}^{\infty} x_n^t : (x_1, x_2, \ldots) \in A_k\right\}.$$

Since A_{k+1} is non-empty, there exists a point $(y_1, y_2, \ldots) \in A_{k+1} \subseteq A_k$, and we obtain

$$\phi_t(\alpha)^t - \sum_{n=1}^{k} M_n^t \leq \sum_{n=k+1}^{\infty} y_n^t$$

for all $t \in (s_k, \infty)$. Therefore,

$$\limsup_{t\to\infty}\left(\phi_t(\alpha)^t - \sum_{n=1}^{k} M_n^t\right)^{1/t} \leq \limsup_{t\to\infty}\left(\sum_{n=k+1}^{\infty} y_n^t\right)^{1/t} = y_{k+1} = M_{k+1}.$$

We have already noted that $M_{k+1} < M'_{k+1}$, which leads to

$$\limsup_{t\to\infty}\left(\phi_t(\alpha)^t - \sum_{n=1}^{k} M_n^t\right)^{1/t} < M'_{k+1}.$$

Hence, there exists $s_{k+1} \in [s_k, \infty)$ such that $\left(\phi_t(\alpha)^t - \sum_{n=1}^{k} M_n^t\right)^{1/t} < M'_{k+1}$ for all $t \in (s_{k+1}, \infty)$.

For any point $(x_1, x_2, \ldots) \in A_k \setminus A_{k+1}$, we have that $M'_{k+1} \leq x_{k+1}$ so that

$$\left(\phi_t(\alpha)^t - \sum_{n=1}^{k} M_n^t\right)^{1/t} < x_{k+1} \leq \left(\sum_{n=k+1}^{\infty} x_n^t\right)^{1/t}.$$

But $(x_1, x_2, \ldots) \in A_k$, which means that $M_n = x_n$ for all $1 \le n \le k$, and it follows that

$$\phi_t(\alpha)^t < \sum_{n=1}^{k} M_n^t + \sum_{n=k+1}^{\infty} x_n^t = \sum_{n=1}^{\infty} x_n^t.$$

Then using (3.8), we have now shown that

$$\min\{\|\mathbf{x}\|_t : \mathbf{x} \in A_k\} < \left(\sum_{n=1}^{\infty} x_n^t\right)^{1/t}$$

for all $(x_1, x_2, \ldots) \in A_k \setminus A_{k+1}$. Finally, we obtain that

$$\phi_t(\alpha) = \min\{\|\mathbf{x}\|_t : \mathbf{x} \in A_k\} = \min\{\|\mathbf{x}\|_t : \mathbf{x} \in A_{k+1}\}$$

for all $t \in (s_{k+1}, \infty)$ verifying the inductive step and completing the proof of our claim.

Next, we observe that $\mathbf{M} = (M_1, M_2, \ldots) \in A \subseteq \mathbb{R}^\infty$, and by definition of $\mathbb{R}^\infty$, there must exist k such that $M_n = 0$ for all $n \ge k$. It follows that $\mathbf{M}$ is the only element in A_k, implying that $\phi_t(\alpha) = \|\mathbf{M}\|_t$ for all $t \in (s_k, \infty)$. By Lemma 3.1, there exists a finite set $\mathcal{X}_0$ such that $\phi_t(\alpha) = \{\|\mathbf{x}\|_t : \mathbf{x} \in \mathcal{X}_0\}$ for all $t \in (a, s_k + 1)$ and we complete the proof by taking $\mathcal{X} = \mathcal{X}_0 \cup \{\mathbf{M}\}$.

□

If a set A can be determined, then our proof reveals a finite process for finding $\mathbf{M}$, even though A is possibly infinite. We are still unaware of a method by which to estimate the number of steps needed to find $\mathbf{M}$.

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